# Parametric correlations of the energy levels of ray-splitting billiards

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Parameter-dependent statistical properties of the spectra of ray-splitting billiards are studied experimentally and theoretically. The autocorrelation functions $c(x)$ and $\tilde{c}(\omega,x)$ of level velocities as well as the generalized conductance $C(0)$ are calculated for two different classically chaotic ray-splitting billiards. Experimentally a modified Sinai ray-splitting billiard is studied consisting of a thin microwave rectangular cavity with two quarter-circle-shaped Teflon inserts. The length of the cavity serves as the experimentally adjustable parameter. For the theoretical estimates of the parametric correlations we compute the quantum spectrum of a scaling triangular ray-splitting billiard. Our experimental and numerical results are compared with each other and with the predictions of random matrix theory.

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A wide class of quantum chaotic systems depends on an external parameter $X$, where $X$ may, for instance, represent the strength of an external field or the shape of the confining perimeter of the quantum system. Upon the rescaling of $X$, the correlation functions of the $X$-dependent energy levels are expected to be universal [1–5]. Of all possible correlation functions, the velocity autocorrelation function takes center stage. It has been studied theoretically in great detail in cases where $X$ is a magnetic field [1–7] or where $X$ characterizes the shape of a quantum billiard [8,9]. The velocity autocorrelation function has also been calculated for random matrix dynamics [10–13]. Given the enormous theoretical interest in the velocity autocorrelation function, it is surprising that experiments addressing its measurement are scarce. We are aware of only four experimental investigations studying the velocity autocorrelation function for (i) a conventional Sinai billiard [14], (ii) the Sinai quartz block [15], (iii) a vibrating plate [16], and (iv) a ray-splitting microwave billiard [9].

In order to evaluate the autocorrelation function of level velocities one should eliminate system-dependent features of the spectra and, instead of the original energy levels $E_i$, consider the unfolded energies

$$\varepsilon_i = N_{av}(E_i), \tag{1}$$

where

$$N_{av}(E) = \int^{E} \rho_{av}(E')\,dE' \tag{2}$$

is the integrated average level density $\rho_{av}(E)$. The parametric motion of the levels has to be unfolded too. This is achieved by introducing the dimensionless parameter

$$x = \int_{X_i}^{X} \sqrt{C(0)}\,dX, \tag{3}$$

where $[X_i, X]$ is the interval of integration,

$$C(0) = \frac{1}{N} \sum_j \left( \frac{\partial \varepsilon_j}{\partial X} \right)^2 \tag{4}$$

is the generalized conductance, and $N$ is the number of energy levels under consideration.

The autocorrelation functions of the level velocities $c(x)$ [1,2,17] and $\tilde{c}(\omega,x)$ [3] are defined as follows:

$$c(x) = \left\langle \frac{\partial \varepsilon_j}{\partial \bar{x}}(\bar{x}) \frac{\partial \varepsilon_j}{\partial \bar{x}}(\bar{x}+x) \right\rangle \tag{5}$$

and

$$\tilde{c}(\omega,x) = \frac{\sum_{i,j} \left\langle \delta[\varepsilon_i(\bar{x}) - \varepsilon_j(\bar{x}+x) - \omega] \frac{\partial \varepsilon_i}{\partial \bar{x}}(\bar{x}) \frac{\partial \varepsilon_j}{\partial \bar{x}}(\bar{x}+x) \right\rangle}{\sum_{i,j} \langle \delta[\varepsilon_i(\bar{x}) - \varepsilon_j(\bar{x}+x) - \omega] \rangle}. \tag{6}$$

In Eq. (5) the average is over the parameter $\bar{x}$ and over all levels. In Eq. (6) the average is only over the parameter $\bar{x}$. In contrast to $c(x)$, which correlates velocities belonging to the same level, $\tilde{c}(\omega,x)$ measures the averaged autocorrelation of velocities separated by a distance $x$ in parameter space and by a distance $\omega$ in energy.

In this paper, we study the velocity autocorrelation functions for two different ray-splitting systems [18–23]: the

 

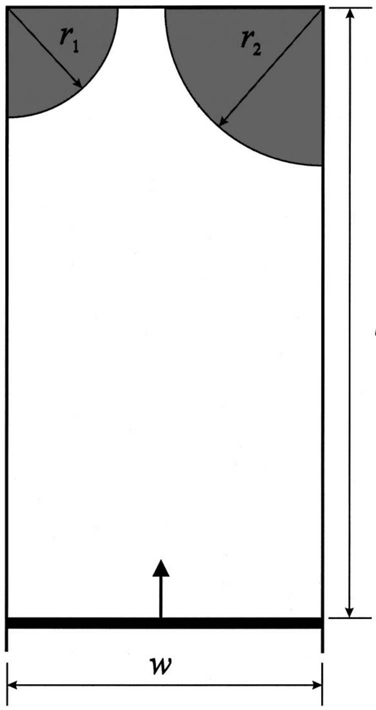


FIG. 1. Sketch of the modified Sinai microwave cavity. The width of the cavity is $w=20$ cm and the length $l$ is varied experimentally between $l_i=43.02$ cm and $l_f=40.52$ cm. Two quarter-circle Teflon disks of radius $r_1=7$ cm and $r_2=10$ cm, respectively, are inserted in the microwave cavity. The height of the inserts is the same as the height of the cavity (0.8 cm).

modified Sinai billiard (MSB) and the triangular step billiard (TSB) [22]. Ray-splitting systems are a new class of chaotic systems in which the underlying classical mechanics is non-Newtonian and nondeterministic [20,22,24]. Ray splitting occurs in many fields of physics, whenever the wavelength is large in comparison with the range over which the potential changes. Ideal model systems for the investigation of ray-splitting phenomena are ray-splitting billiards [18,20,22,23] and microwave cavities with dielectric inserts [9,19,25,26].

The modified Sinai microwave cavity consists of a thin cavity of dimensions $h=0.8$ cm (height) and $w=20$ cm (width) with two quarter-circle Teflon inserts of radius $r_1=7$ cm and $r_2=10$ cm, respectively (see Fig. 1). For frequencies $\nu$ less than a cut-off frequency $\nu_c=c/(2hn)$, where $c$ is the speed of light and $n$ is the index of refraction of the dielectric insert, the electrodynamics of a thin microwave cavity can be described by the Helmholtz equation [27]. It is equivalent to the Schrödinger equation in a two-dimensional quantum billiard [28–30]. For a Teflon-loaded microwave cavity the index of refraction is $n\simeq 1.44$ [31] and the cut-off frequency is $\nu_c\simeq 13$ GHz.

The length of the microwave cavity was chosen as an experimentally adjustable parameter to generate level dynamics. We have changed the length $l$ of the microwave cavity in the range from $l_i=43.02$ cm to $l_f=40.52$ cm in 50 steps of 0.05 cm, and measured resonance frequencies $\nu_j$, $j=10, \ldots ,356$ as a function of the parameter $l$ for the frequency range from 2 to 10.4 GHz. The cavity's spectra were

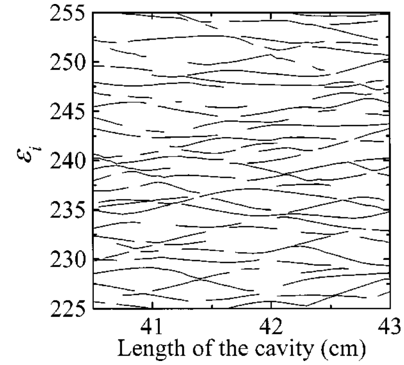


FIG. 2. Unfolded spectrum of the modified Sinai billiard for the eigenenergies $\varepsilon_i$, $i=225$–255.

measured using a frequency step of 0.2 MHz, which we have found to be satisfactory for revealing all the details of the spectra. The eigenfrequencies with counting index $j$ larger than 356 ($\nu>10.4$ GHz) were not measured. Although rather short antennae (length 2.5 mm) were used to allow for only a small perturbation of the cavity's field, its quality factor $Q$ was rather low ($Q\approx 10^3$ for $\nu>10$ GHz) and therefore the resolution of the resonances measured in this region was poor. The measured resonance frequencies are converted to ''eigenergies'' of the two-dimensional microwave billiard according to the formula [28] $E_j(l)=k_j^2(l)$, where $k_j(l)=2\pi\nu_j(l)/c$ are the wave numbers. A typical set of unfolded energy levels of the modified Sinai microwave cavity, as a function of $l$, is shown in Fig. 2. The expression for the integrated average density of levels $N_{av}(E)$ for the MSB required for the unfolding, was obtained from formula (2.22) in Ref. [26]. Figure 2 shows that the level dynamics are irregular.

The ray-splitting triangular step billiard [22] considered in this paper, is shown in Fig. 3. It is an isosceles triangle with a top angle of $6\pi/20$. The bisector of the top angle is a ray-splitting boundary dividing the triangle into two equal parts with two different potentials, $V=0$ and $V=V_0>0$. It is proved analytically that there are no elliptic islands in the classical phase space of the TSB [22]. There is also numerical evidence that the TSB is ergodic [22]. The stationary

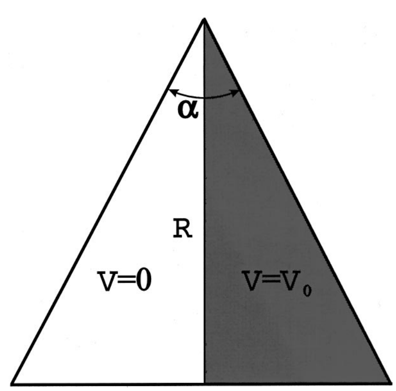


FIG. 3. The isosceles triangular step billiard. The ray-splitting boundary $R$ separates the two domains of the billiard with $V=0$ and $V=V_0>0$. The angle is $\alpha=6\pi/20$.

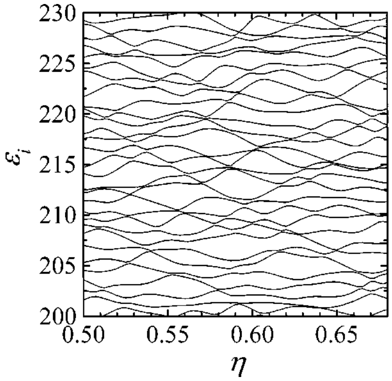


FIG. 4. Unfolded spectrum of the triangular step billiard for the eigenenergies $\varepsilon_i$, $i=200$–230. The scaling parameter $\eta$ is defined as $\eta=V_0/E$ (Ref. [22]).

Schrödinger equation for the TSB was solved numerically to obtain 400 eigenenergies. The level dynamics was revealed by solving the Schrödinger equation for 73 different values of the scaling parameter $\eta=V_0/E$ (Fig. 4). The spectra of the TSB were unfolded by means of the integrated average density of levels $N_{av}(\eta)$ taken from Ref. [22].

The main point of this paper is to study the universality conjecture by investigating parameter-dependent correlation functions for systems that depend on different control parameters in substantially different ways. In the MSB the external parameter $X$ is the length of the cavity while in the TSB the parameter $X$ represents the potential scaling parameter $\eta$ which, in an electromagnetic context, would be equivalent to the variation of the dielectric constant while preserving the shape and size of the cavity and the dielectric. From an experimental point of view, a significant variation of the dielectric constant inside a microwave cavity would pose a very difficult problem.

Before performing any calculations of the autocorrelation functions of level velocities one should estimate how many of the low-lying levels may possibly display nonuniversal behavior. Two different measures were used for this purpose: the nearest-neighbor spacing (NNS) distribution and the distribution of parametric velocities $\partial\varepsilon/\partial x$. For time-reversal invariant classically chaotic systems, which correspond to the Gaussian orthogonal ensemble (GOE) in random matrix theory (RMT) (the case considered here), the NNS distribution should resemble a Wigner distribution [29], while the distribution of parametric velocities should be Gaussian [32].

For the first 165 levels, careful analysis of the NNS distributions for the Sinai billiard reveals a significant deviation of these distributions from the Wigner surmise. The NNS distributions were fitted to the Brody formula [33]

$$P_q(s)=b_q s^q e^{-a_q s^{1+q}} \quad \text{with}$$
$$b_q=(1+q)a_q, \quad a_q=\left[\Gamma\left(\frac{2+q}{1+q}\right)\right]^{1+q}. \tag{7}$$

For $q=0$, the Brody formula becomes the Poissonian distribution $P_0(s)=e^{-s}$, which describes integrable systems; for $q=1$ it assumes the form of a Wignerian distribution $P_1(s)=(\pi/2)se^{-(\pi/4)s^2}$, which describes quantum chaotic systems [29]. Thus the parameter $q$ in the Brody formula can be used as a convenient measure of the degree of chaoticity.

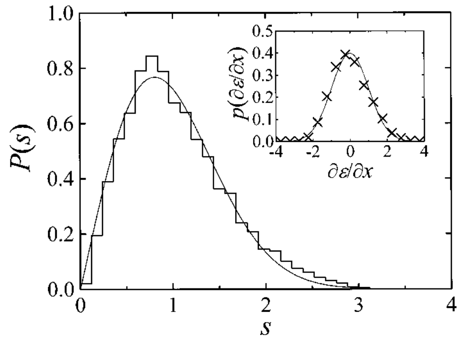


FIG. 5. Nearest-neighbor spacing distribution for the MSB for level numbers $N=225$–356. The NNS distribution was fitted to the Brody formula (see the text) with a Brody parameter of $q=1.03\pm0.05$. The inset shows the parametric velocity distribution for the MSB compared to a Gaussian $P(\xi)=(1/\sqrt{2\pi})\exp(-\xi^2/2)$.

For the first 165 levels of the MSB we find $q=0.72\pm0.05$. The NNS distribution for higher levels ($j=225,\ldots,356$) of the MSB is shown in Fig. 5. In this case we find a Brody parameter of $q=1.03\pm0.05$, and the NNS distribution is very close to the Wignerian distribution. The inset in Fig. 5 shows the distribution of parametric velocities constructed from the energy levels with counting index $j=225,\ldots,356$ compared with the theoretically expected distribution. We obtain very good agreement between theory and experiment. We mention that for the first 165 levels of the MSB the distribution of velocities (not shown) has a pronounced non-Gaussian character.

In the case of the triangular step billiard, both the NNS distributions for the lower levels $j=1,\ldots,100$ (not shown) and the higher levels $j=150,\ldots,400$ (see Fig. 6) are well described by Wignerian distributions. We obtained $q=0.97\pm0.04$ for the levels $j=1,\ldots,100$ and $q=0.98\pm0.04$ for the levels $j=150,\ldots,400$, respectively. However, the distribution of parametric velocities for the lower levels $j=1,\ldots,100$ is non-Gaussian. The distribution of velocities

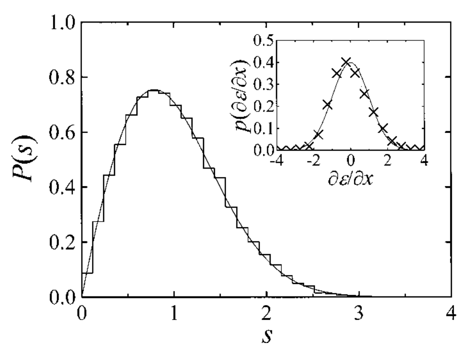


FIG. 6. Nearest-neighbor spacing distribution for the TSB for level numbers $N=150$–400. The NNS distribution was fitted to the Brody formula (see the text) with a Brody parameter of $q=0.98\pm0.04$. The inset shows the parametric velocity distribution for the TSB compared to a Gaussian $P(\xi)=(1/\sqrt{2\pi})\exp(-\xi^2/2)$.

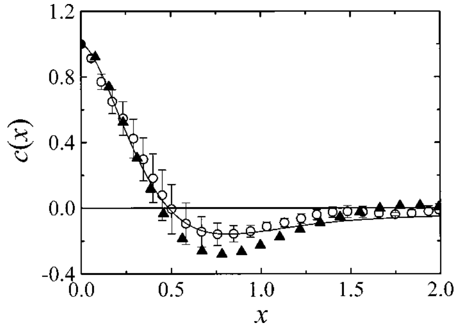


FIG. 7. Velocity autocorrelation function $c(x)$ for the modified Sinai billiard (circles) and the triangular step billiard (triangles) compared to the predictions of RMT (full line).

for levels $j=150, \ldots, 400$ is presented in the inset of Fig. 6. This distribution is very close to a Gaussian. Taking into account NNS and parametric velocity distributions, we decided to use level numbers $225, \ldots, 356$ of the MSB and level numbers $150, \ldots, 400$ of the TSB in the calculations of velocity autocorrelation functions discussed below.

In Fig. 7 we show the parametric velocity autocorrelation function $c(x)$ for the modified Sinai billiard and for the triangular step billiard compared to the predictions of RMT for GOE [34]. The numerical simulations of the parameter-dependent correlation functions within RMT were made with the help of the following model [8]:

$$H(X)=H_1\sin(X)+H_2\cos(X), \tag{8}$$

where $H_1, H_2$ are two $500\times 500$ matrices that are members of the GOE. In our calculations, the parameter $X$ was estimated in 1001 points uniformly distributed in the interval $(0,\pi/8)$. For each value of the parameter $X$ we ran 99 realizations of $H_1$ and $H_2$. The eigenvalues obtained from the diagonalization of the Hamiltonian $H(X)$ were unfolded by using the integrated average eigenvalue density that was derived via Eq. (2) from the average density of the eigenvalues of GOE matrices [34].

For the Sinai ray-splitting billiard, a good overall agreement with the results predicted by RMT is observed. For small values of the parameter $x$ ($x<0.2$) the experimental data shows a small downward deviation from the RMT results. For larger values of $x$ it follows the theory well, until the minimum on the curve is reached. For still larger $x$, the experimental curve diverges from the theoretical prediction in the upward direction. The autocorrelation function for the TSB agrees with the theory for small values of the parameter but for intermediate values of $x$ ($0.4<x<1.2$) it deviates significantly in the downward direction. This deviation has been previously observed for other chaotic systems [15]. For $x>1.5$ the autocorrelation function $c(x)$ shows deviations in the upward direction.

For a more thorough examination of the two systems' parametric correlations, we computed the velocity autocorrelation functions $\tilde{c}(\omega,x)$. Figure 8 shows the velocity autocorrelator $\tilde{c}(\omega,x)$ for $\omega=0.25$, 0.5, and 1.0 for the MSB [panel (a)] and the TSB [panel (b)]. In our calculations we averaged over an energy interval $\delta=0.03$ [3]. The calculations are compared to the RMT results obtained from the model Hamiltonian (8). For the modified Sinai billiard, the overall agreement with the RMT predictions is rather good but, especially for $\omega=0.25$ and $x\simeq 0.11$, the experimental peak exceeds the RMT result. For the TSB, we find good agreement with RMT predictions. For small $x$ the data lie somewhat lower than predicted by the theoretical curve. With increasing $x$ they reach peak values that exceed the RMT prediction. The data for the TSB also decay slightly faster than the corresponding RMT prediction.

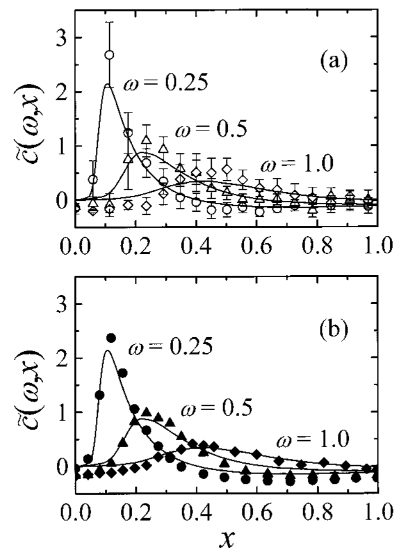


FIG. 8. Velocity autocorrelation function $\tilde{c}(\omega,x)$ for the modified Sinai billiard [panel (a)] and the triangular step billiard [panel (b)] compared to the predictions of RMT (full lines) for the three values of the parameter $\omega$: $\omega=0.25$ (circles), $\omega=0.5$ (triangles), and $\omega=1$ (diamonds).

For two-dimensional billiards in the semiclassical regime, the system-dependent generalized conductance $C(0)$ was found to scale with energy according to $C(0)\sim\varepsilon^{3/2}$ [8,9,32]. This property was tested for both the TSB and the MSB billiards (Fig. 9). Each point on the curves in Fig. 9 was obtained by averaging over 10 neighboring levels for the TSB (Fig. 9) and over 20 neighboring levels for the MSB

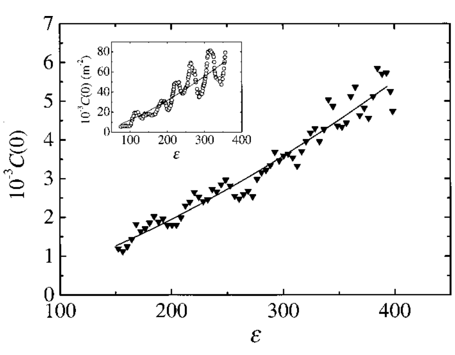


FIG. 9. Generalized conductance $C(0)$ for the triangular step billiard. The full line is the least-squares fit: $C(0)=a\varepsilon^b$, with $a=0.685\pm 0.201$ and $b=1.50\pm 0.05$. The inset shows $C(0)$ for the modified Sinai billiard. The full line is the least-squares fit: $C(0)=a\varepsilon^b$, with $a=19.9\ \mathrm{m}^{-2}\pm 13.2\ \mathrm{m}^{-2}$ and $b=1.39\pm 0.13$.

(inset in Fig. 9). In both cases the data were also averaged over the full range of the parameter $X$. For these data, a least-squares fit to the function

$$C(0)=a\varepsilon^{b} \tag{9}$$

was performed with the two fit parameters $a$ and $b$. From the fit for the TSB, we obtained the parameter values $a=0.685\pm0.201$ and $b=1.50\pm0.05$, respectively. Thus the exponent $b$ is in excellent agreement with the theory. The inset of Fig. 9 shows that for the MSB, $C(0)$ exhibits strong oscillations possibly connected with the presence of bouncing ball orbits [35]. Due to this oscillatory behavior, we decided to extend the range of fitting to the lower energies corresponding to $N=75$–356. For the modified Sinai billiard, we obtained $a=19.9\ \mathrm{m}^{-2}\pm13.2\ \mathrm{m}^{-2}$ and $b=1.39\pm0.13$. The parameter $b$ coincides with the expected value 3/2 within the error limits.

In summary, we presented a detailed experimental and theoretical study of the velocity autocorrelation functions $c(x)$ and $\tilde{c}(\omega,x)$ for two different ray-splitting systems: the modified Sinai billiard and the triangular step billiard. For the MSB we found good agreement of the estimated $c(x)$ with the RMT prediction. However, the autocorrelation functions $\tilde{c}(\omega,x)$ exceed the RMT results, though they are still within the experimental error. For the TSB, the correlation function $c(x)$ deviates significantly in the downward direction for intermediate values of $x$ $(0.4<x<1.2)$. Since this deviation has been observed previously for other chaotic systems [9,15] our results add substance to the suspicion that not all statistical properties of classically chaotic systems may be accurately described by RMT. In contrast to $c(x)$, the autocorrelation functions $\tilde{c}(\omega,x)$ calculated for the TSB, are in good agreement with the RMT predictions. In both cases, however, the scaling of the universal conductance $C(0)$ was close to the theoretical prediction $C(0)\sim\varepsilon^{3/2}$.

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